The Case for Hyperbolic Theories of Dissipation in Relativistic Fluids

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In this paper we highlight the fact that the physical content of hyperbolic theories of relativistic dissipative fluids is, in general, much broader than that of the parabolic ones. This is substantiated by presenting an ample range of dissipative fluids whose behavior noticeably departs from Navier-Stokes' .

I. INTRODUCTION

As is well-known causal theories of dissipative fluids, both relativistic [1] and non-relativistic [2], [3] were set up to get rid of some undesirable effects connected to the conventional Eckart-type theories [4]. The advantages of the causal theories are the following [5]: (i) For stable fluid configurations the dissipative signals propagate causally. (ii) Unlike Eckart-type's theories, there is no generic short-wavelength secular instability in causal theories. (iii) Even for rotating fluids, the perturbations have a well-posed initial value problem.

The central idea in causal-type theories (also called "hyperbolic") is to extend the space of variables of conventional theories by incorporating in it the dissipative quantities (i.e., heat fluxes, particle currents, shear and bulk stresses, etc.) of the phenomenon under consideration. Hence these quantities are treated on the same footing as the "conserved" variables (energy density, particle numbers, etc.). The price to be paid is a more involved theory with a larger number of variables and parameters. In the approach of Israel [1.a-c] as well as in the one by Pavón *et al* [1.d], by combining the conservation of the stress-energy tensor and the conservation of the particle current density with the Gibbs equation (which has extra terms due to the enlargement of the space of variables) an expression is obtained for the entropy production. The transport equations for the dissipative fluxes follow from the latter by demanding that the aforesaid production be nonnegative. A distinguishing feature of these equations as compared with the corresponding ones in Eckart-type theories is that the former contain a "relaxational term" -the convective time derivative of the corresponding flux multiplied by a parameter with the dimension of time. Similar equations can be obtained in the extended thermodynamics approach of Müller and Ruggeri [2.b]. The transport equations can be understood as evolution equations for the dissipative variables as they (being "hyperbolic" in nature) describe how these fluxes evolve from an initial arbitrary state to a final steady one. On the contrary, the Eckart-type theories are fit for quasi-steady states only, and they are usually referred to as "parabolic" or "quasi-stationary theories".

The aforesaid time parameter τ is usually interpreted as the relaxation time of the dissipative process. For instance, in heat conduction it is the time taken by the heat flux to attain its steady value since the establishment of a temperature gradient across the system. It reflects the fact that the effect (the heat flux or any other dissipative phenomenon) lags behind from its cause (the temperature gradient or any other thermodynamic "force" acting on the system). This is directly linked to the "second sound" -a well known phenomenon in laboratory physics whose justification can be found within hyperbolic theories. We will turn to this topic below. Even in the steady regime the descriptions offered by causal and acausal theories do not necessarily coincide. The differences between them in such a situation arise from (i) the presence of τ in terms that couple the vorticity to the heat flux and shear stresses. These may be large even in steady states (e.g. rotating stars). There are also other acceleration coupling terms to bulk and shear stresses and heat flux -see for instance [5.a]. The coefficients for these vanish in parabolic theories, and they could be large even in the steady state. (ii) From the convective part of the time derivative (which are not negligible in the presence of large spatial gradients). (iii) From modifications in the equations of state due to the presence of dissipative fluxes. However, it is precisely before the establishment of the steady regime that both types of theories (hyperbolic and parabolic) can differ more importantly. Therefore if one wishes to study a dissipative process for times shorter than τ, it is mandatory to resort to a hyperbolic theory which is a more accurate macroscopic approxima-

tion to the underlying kinetic description. Only for times longer than τ it is permissible to go to a parabolic one (provided that the spatial gradients are not so large that the convective part of the time derivative does not become important, and that the fluxes and coupling terms remain safely small). All this has long been well-known.

Recently, however, it has been argued that the physical content of causal (hyperbolic) and stationary (parabolic) theories *is the same* [6], [7]. Such a claim is based on the study of simple dissipative fluids under the assumption that its relaxation time is of the same order than the colision time between particles -something that, as we shall show is far from being always the case. Moreover in [7] it is asserted that such a study is *very general and should apply to essentially any theory of fluids (including those describing superfluids, mixtures of different kinds of fluids, etc.)*; and that *the meaningful differences between the causal theories and the non-causal Navier-Stokes theory can not be observed. The complicated dynamical structure of the causal theories is necessary to insure that the fluid evolves in a causal and stable way. But this rich dynamical structure is unobservable, since the physical states of a fluid always evolve in a way that is also well described by the Navier-Stokes expressions for the stress-energy tensor, etc.* These assertions (as well as some others of the like in [6] and [7]) are misleading (to say the least) and likely to promt confusion. The purpose of [6] was to justify some class of relativistic, hyperbolic theories of dissipative fluids. This was achieved for simple fluids under the assumptions that (i) the departure of these fluids from the Fourier-Navier-Stokes behavior is negligible small, and (ii) the relaxation times are extremly short on macroscopic scales. Notice that in [6] it is not explictly denied that on examining a typical FNS (Fourier-Navier-Stokes) fluid in certain regimes, or with sufficient accuracy, there will be found non-FNS behavior, which could be satisfactorily described by some hyperbolic theory. However, it is explicitly suggested that *it may happen that the hyperbolic theories have no useful physical consequences at all.*

The aim of the present paper is to explicitly show that the physical content of hyperbolic (causal) transport theories is much broader than the corresponding one of parabolic (quasi-stationary) theories. We will do this by presenting a series of well-known physical results in an ample range of dissipative systems including simple fluids. A criticism in [6] and [7] of the hyperbolic theories is that they contain a number of *arbitrary* parameters. We would like to emphasize that this is not the case. In fact these parameters can be determined provided the state equations of the fluid under consideration are known. From these such parameters can be calculated either by kinetic theory or fluctuation theory. Israel and Stewart computed them for a quantum gas using Grad's 14-moment approximation -see the set of equations (7.8) in [1.b]. For an example in radiative fluids, in which the coefficients connecting the heat flux with the gradient of the dissipative stresses are computed, see [8]. Further,

a clear conceptual difference between the physical content of both set of theories is the following one. With the help of Einstein's theory of thermodynamic fluctuations hyperbolic theories are capable of deriving the transport coefficients (heat conductivity κ, bulk ζ and shear η stresses, etc) from the fluctuations, because they are based on an extended entropy (which comprises the dissipative fluxes), at variance with parabolic theories. This has been done, for instance, for radiative fluids [9]. This however cannot be done in the case of parabolic theories, which necessarily have to be supplemented with microscopic expressions to obtain these transport coefficients.

Section II recalls the concept of relaxation time. Section III brings about seven examples in the non-relativistic context showing that the physical content of hyperbolic theories is much wider than that of the parabolic ones. These include examples involving heat and/or electric conduction in solids. Note that there is nothing wrong in that because it is the electron and the phonon gases in the solid that are responsible for such transport processes within it [10]. Thereby it is meaningful to apply, in this regard, fluid theories to solids. Section IV deals with four fully relativistic examples. In Section V our main conclusions and outlooks are summarized.

II. THE RELAXATION TIME

Before going any further it is expedient to say a few words about τ. The relaxation time is a key quantity in hyperbolic systems with a distinct physical meaning as made explicit before. It is connected to the mean collisional time of the fluid particles t_{col}, sometimes erroneously identified with it. In principle they are different since τ is a macroscopic time, although in some instances it may correspond to a few t_{col}. It is more appropriate to interpret τ as the time taken by the corresponding dissipative flux to relax to its steady value. We believe that at the root of the confusion raised by [6] and [7] is, among other things, the assumption that τ and t_{col} are *always* of the same order. In fact one of the main arguments in Ref [7] heavily rests on the assumption that the dimensionless quantity $\Gamma \equiv (\tau c_s/L)^2$ is always negligible. Likewise it is crucially needed for the bounds established by equations (3.22) and (3.23) in [6] to hold. (Here c_s stands for the isentropic speed of sound in the fluid under consideration and L the characteristic length of the system). That assumption would be right if τ were always comparable to t_{col} and L always "large", but as we shall see there are important situations in which $\tau \gg t_{col}$, and L "small" although still large enough to justify a macroscopic description. For tiny semiconductor pieces of about 10^{-4} cm in size, used in common electronic devices submitted to high electric fields, the above dimensionless combination (with $\tau \sim 10^{-10}$ sec, $c_s \sim 10^7$ cm/sec [11]) can eas-

ily be of the order of unity. In ultrasound propagation as well as light-scattering experiments in gases and neutron-scattering in liquids the relevant length is no longer the system size, but the wavelenght λ which is usually much smaller than L [3], [27], [13]. Because of this, hyperbolic theories may bear some importance in the study of nanoparticles and quantum dots.

Likewise in polymeric fluids relaxation times are related to the internal configurational degres of freedom and so much longer than t_{col} (in fact they are in the range of the minutes), and $c_s \sim 10^5$ cm/sec, hence it follows $\Gamma \sim \mathcal{O}(1)$. In the degenerate core of aged stars the thermal relaxation time can be as high as 1 second [14]. Assuming the radius of the core of about $10^{-2} R_{SUN}$ one has $\Gamma \sim \mathcal{O}(1)$ again. Think for instance of some syrup fluid flowing under a shear stress, and imagine that the imposed shear is suddenly suppressed. This liquid will come to rest only after a much longer time (τ) than the collision time between its constituent particles has elapsed. Many other examples could be added but we do not mean to be exhaustive.

In transport theory when the mean free-path of the particles is much smaller than the macroscopic length scale of interest, the gas is in a state of small deviations from local thermal equilibrium (LTE) and the usual continuum assumptions are appropriate. In the gas dynamical case the local thermal equilibrium equations are the FNS system which are parabolic in nature. When the mean free-path is of order of the macroscopic length scale the usual continuum description is no longer valid and the more fundamental kinetic approach must be considered. Under rarefied situations (i.e., limiting to the one particle distribution function) the kinetic description is embodied in the Boltzmann Transport Equation (BTE). Solution techniques for this equation fall in two main groups: (i) direct, particle methods, and (ii) extended continuum approaches (e.g. based on moment methods). In general, direct particle methods (e.g. Monte Carlo) are capable of obtaining solutions under general conditions, whereas extended continuum approaches tend to be restricted to some definite ranges of the parameters. However, for transitional fluxes ranging from very large mean free-paths (where only a direct particle method could be applicable) and very small mean free-path (where a usual continuum description is warranted) to utilize direct particle methods would lead to prohibitive computational costs. Under these conditions extended continuum methods might be computationally much more efficient. In the extended thermodynamics framework the moment equations obtained from the Boltzmann transport equation are closed by utilizing the entropy principle (Müller *et al* [2.b]; Jou *et al* [3]) or by assuming the maximum entropy distribution function (i.e., a special form for the distribution function which maximizes entropy subjected to the given moments [3]). The resulting set of transport equations form a quasi-linear, symmetric and hyperbolic system of conservation laws (in a neighbourhood of LTE) which admits a supplementary conservation law in the form of entropy balance. From the mathematical viewpoint this system has desirable properties. The obtained mathematical model is able to describe efficiently the regimes that lie in between the collisionless and the collisional regimes, i.e., the transitional one. In fact these models capture the usual continuum regime when the mean free-path is much smaller than the macroscopic length scale while in the transition regime provide values for the physical quantities that are consistent with a positive definite distribution function, and therefore are physically more satisfactory. These models are potentially very useful for constructing hybrid continuum/kinetic schemes for simulation purposes (i.e., numerical schemes where the kinetic approach, which is computationally very expensive, is used only where it is required, that is, in the collisionless case, where continuum equations are used in the near LTE, and extended continuum models in the transition regions) [15]. Therefore, hyperbolic theories broaden the domain of continuum models and bridge the gap between collisional and collisonless regimes. Then the scope of hyperbolic theories is to describe the transitional regimes and it is meaningless to restrict them just to the collision dominated case.

III. NON-RELATIVISTIC EXAMPLES

1. *Heat flux and shear stress in ideal gases*
   Let us consider an ideal gas subjected to a temperature gradient and a shear stress. It has been established with the help of Grad's thirteen moment approximation [16] that the the relaxation times for the heat flux and the shear pressure are given by $\tau_q = 3/(2nm\gamma)$ and $\tau_s = 2\tau_q/3$, respectively. Here n stands for the particle number density, m the molecular mass, and γ a positive constant related to the interaction cross section -see §3.3 in Ref [3]. It is obvious that in a collision dominated regime n will be large and hence both relaxation times will not be much longer than t_{col}. In the opposite extreme (Knudsen regime) the continuum approximation breaks down. However, for intermediate situations (where hyperbolic theories apply) both τ_q and τ_s can be substantially larger than t_{col}.

2. *Second sound in superfluids and solids*
   The second sound was first discovered by Peshkov [17] in Helium II in the mid-forties, and later confirmed, also in solids, by a number of experimentalists (see, e.g. [18], [19] and [20]). It is related to the propagation of the heat signal in solids not at the phase speed predicted by Fourier's law,

$$v_{ph} = \sqrt{2\chi\omega}$$

(which diverges as the frequency of the signal $\omega \to \infty$), with

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$$\chi \equiv \kappa/(\rho c_v)\,,$$

where κ, ρ and c_v stand for the thermal heat conductivity, the mass density and the specific heat at constant volume of the solid, respectively, but at the speed

$$v_{ph} = \frac{\sqrt{2\chi\omega}}{\sqrt{\tau\omega + \sqrt{1 + \tau^2\omega^2}}},$$

which remains finite at high frequencies. This result straightforwardly follows from combining the Maxwell-Cattaneo equation,

$$\vec{q} + \tau\frac{\partial\vec{q}}{\partial t} = -\kappa\nabla T\,,$$

typical of hyperbolic theories, with the equation for the balance of the energy in a rigid solid,

$$\rho\dot{\epsilon} = -\nabla\cdot\vec{q}$$

-see §6.1 of [3] for details. At low frequencies (i.e., $\tau\omega \ll 1$) the phase velocity of the hyperbolic theory reduces to the one predicted by the Fourier's law. However, outside that range both equations significantly differ from one another; and in the high frequency limit ($\tau\omega \gg 1$) the former reduces to

$$U = \sqrt{\chi/\tau}\,,$$

which is the propagation speed of thermal pulses - see §6.2 in [3]. As mentioned above all these results have been confirmed experimentally -see [20] and [21] for a review. Notice that if one makes the unphysical assumption implied by equating τ to zero (which corresponds to the Fourier's law) the latter expression diverges.

Closely connected to the above discussion is the response of material sample to an instantaneous heat pulse (activated by means of a laser beam, or an electric discharge). An elegant treatment of the problem can be implemented by adding a delta-like term to the right hand side of the energy balance equation. This when combined with the Maxwell-Cattaneo equation leads to a second order differential equation in terms of the associated Green function, and one obtains for the temperature response to the heat pulse the expression

$$\Delta T(x,t) = \frac{g_0}{4\kappa}\exp\left(-\frac{t}{2\tau}\right)\left[2I_0(\xi)U\tau\delta(Ut - \mid x\mid)\right.$$
$$\left. + \left(I_0(\xi) + \frac{t}{\tau\xi}I_1(\xi)\right)H(Ut - \mid x\mid)\right]\,,$$

(where, for simplicity, we have restricted ourselves to one spatial dimension) while the temperature

response corresponding to the parabolic theory (Fourier's law) is

$$\Delta T(x,t) = \frac{g_0}{2\sqrt{\pi\chi t}}\exp\left(-\frac{x^2}{4\chi t}\right)\,.$$

Here g_0 is the energy supplied to the sample, $\xi \equiv \frac{1}{2}\sqrt{(t/\tau)^2 - (x/U\tau)^2}$, $I_0(\xi)$ and $I_1(\xi)$ are related to the Bessel functions of the n kind by $I_n(\xi) = -(-i)^n J_n(i\xi)$, and $H(U - t \mid x \mid)$ is the Heaviside unit step function.

There are important differences between the predictions of the conventional parabolic theory and the causal theory. (i) In the former, the response to a heat pulse is instantaneously felt in all the sample. In the latter, by contrast, it vanishes for $x > Ut$. (ii) Two distinct contributions enter $\Delta T_{hyperbolic}$. On the one hand there is the term with $\delta(Ut - x)$ which reflects the initial pulse damped by an exponential factor; on the other hand there is, in addition, the Heaviside term which corresponds to the wake. The latter after a sufficiently long time reduces to the conventional diffusion solution. Such a structure has been observed in the propagation of heat pulses in solids at low temperatures (see Figure 1 for a schematic representation of the experimental results) and in the structure of X-ray peaks from X-ray bursters in star explosions -see point 1 of Section IV below. (iii) The hyperbolic description predicts the presence of a precursor peak which has no counterpart in the parabolic theory. Although both situations are characterized by the same total energy, the distribution of the thermal energy differs from one another. While in the acausal description the energy is roughly evenly distributed in the whole spatial region, in the causal description, by contrast, there is a sharp concentration of energy associated to the precursor peak.

3. *Ultrasond waves and light-scattering in ideal monoatomic gases*
In the low frequency regime (i.e., $\tau\omega \ll 1$) the sound propagation as well as light-scattering in ideal gases, with a heat flux and shear stress, are well described by the parabolic FNS theory of dissipative phenomena. However, as ω is increased the limitations of this theory begin to be felt (in fact as soon as $\tau\omega \simeq 0.5$ notorious discrepancies with respect to the FNS theory appear); and in the high frequency regime ($\tau\omega \gg 1$) the disagreements with experimental results become wider. Aside from predicting an unbounded sound speed as $\omega \to \infty$ it fails to foretell the existence of an additional wave and yields erroneous values for the absorption coefficient. Things fare differently for hyperbolic theories. These predict constant phase speed for both waves, $1.64\,c_s$ and $0.72\,c_s$ with $c_s = \sqrt{5\,k_B T/(3m)}$, as $\omega \to \infty$ [22], [23]. All these results show good

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agreement with the well-known experimental data of Meyer and Sessler [24] and Greenspan [25] about ultrasound waves.

Table I contrasts the predictions of the FNS and hyperbolic theories [26] with the corresponding experimental results in terms of the ratio between the mean free path l of the moleculas and the wavelength λ of the soundwave.

Light scattering in dilute ideal monoatomic gases has been studied in detail by Weiss and Müller [27]. These authors find good agreement of the hyperbolic theory with the experimental data of Clark [28] (see figures 7.1 through 7.3 for the dynamic form factor in [27]) while the parabolic FNS theory fails hopelessly.

4. *Electric circuits*
   Simpler and more familiar examples can be found in the province of electric circuits. To fix ideas consider a resistor and an inductance connected to a source $\mathcal{E}(t)$. The intensity in the resulting circuit is governed by $\mathcal{E}(t) = RI(t) + L\,dI(t)/dt$. Note that if the source were suddenly turned off, the current will decay in a characteristic (relaxation) time given by $\tau_e = L/R$. It is apparent that by suitably choosing R and L the aforesaid time can be made much longer than the collisional time of the electrons in the circuit.

5. *Thermohaline instability*
   Imagine a layer of warm salt water above a layer of fresh cold water. This system will be dynamically stable against sinking so long the warmth of the salt water is high enough to reduce its specific weight to below that of the fresh water. However, as the upper layer cools off, its density augments and eventually small blobs of salty water will sink down. This kind of instability may also occur in stars [29]. Let ΔT be the temperature difference between a generic blob of salt water and its surrounding and

$$\tau_d = \frac{\chi \rho^2 a^2 c_p}{4 a_B c T^3}$$

the adjustment time where χ denotes the mean absorption coeffcient, ρ the mass density of the blob, a the radius assuming the blob spherical, a_B the Boltzmann radiation constant, and c_p the specific heat at constant pressure. If the evolution of ΔT is analyzed using the Fourier law for the the conductive heat transport $\vec{q} = -\kappa \nabla T$, one obtains an exponential decrease $\Delta T = \Delta T(t=0)exp(-t/\tau_d)$, and the blob simply sinks in the water below it. However, if the Maxwell-Cattaneo equation is used in place of Fourier's one, it follows

$$\Delta T(t) = \Delta T(t=0)e^{-x}\left\{\cos(\omega x)\right.$$

$$\left. -\left(1 + \frac{2\beta}{\Delta T(0)\tau_d}\right)\frac{\sin(\omega x)}{\omega}\right\}$$

with $x \equiv t/2\tau$, $\omega \equiv \sqrt{4\tau/\tau_d - 1}$ and

$$\beta \equiv \int_{-\infty}^{0} \Delta T(t)exp(t/\tau)dt$$

and the blob undergoes an oscillatory motion -see [30] for details. So for times shorter than the effective time for relaxation into diffusion the behavior of the blobs differ drastically from that predicted by the parabolic theory.

6. *Polymeric fluids*
   Simple models of dilute polymeric solutions accounting nicely for the observed dynamics of these fluids are, among others, those of Rouse and Zimm [31]. In these the polymer is assimilated to a chain of beads connected by Hookean springs of constant K, and the viscous pressure tensor corresponding to the mode m is governed by the Maxwell-Cattaeno type equation

$$P_{ik}^{(m)} + \tau^{(m)}\frac{\partial P_{ik}^{(m)}}{\partial t} = -\frac{1}{2}\eta^{(m)}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$$

with $\eta^{(m)}$ the viscosity coefficient and $\tau^{(m)} = \xi/(2Ka^{(m)})$ the relaxation time of that mode. Here ξ is the friction coefficient, $a^{(m)} = 4\sin(\pi m/2N)$ the eigenvalues of the Rouse matrices and N the number of monomers. It is apparent that $\tau^{(m)}$ can be made as large as desired just by increasing this number. For large N one has $\tau^{(m)} \sim N^2$. The above equation for $P_{ik}^{(m)}$ can be easily derived in the framework of hyperbolic theories - see for instance Chapter 7 in Ref [3].

7. *Carrier transport in semiconductor devices*
   Enhanced functional integration in modern electron devices requires an accurate modelling of charge carrier transport in doped semiconductors in order to describe high field phenomena such as hot electrons, impact ionization, etc [32]. The standard drift-diffusion equations [33] cannot describe adequately high field and submicron phenomena because they hold only in the limit of weak electric field and short mean free-paths [34]. The ultimate description (at least at the semiclassical level) is to find a direct solution to the semiclassical BTE [34] by numerical techniques such as Monte Carlo simulation [35]. Such an approach, however, requires very long computing times ad is therefore unsuitable for engineering design applications. An intermediate approach is to write a relatively small number of moment equations of the BTE, closing the system of equations by physical and mathematical ansatz (the closure assumption) which must

then be checked by Monte Carlo simulation. The obtained macroscopic models, with the transport parameters (relaxation times, heat conduction coefficient, etc.) obtained by Monte Carlo verified physical and mathematical ansatz, are called hydrodynamical models and their range of validity is somehow in between the drift-diffusion regime and the kinetic one.

Lately Anile *et al* [36], [11] have introduced an extended hydrodynamical model in which the closure of the moment equations is achieved by imposing the entropy principle of extended thermodynamics [2.b]. In the case of silicon semiconductors, there are several distinct time scales, the energy relaxation time τ_w, the momentum relaxation time τ_p, the energy flux relaxation time τ_q, etc. Monte Carlo simulation shows that $\tau_w >> \tau_p > \tau_q$, due to the predominance of elastic collisions, and this justifies the existence of a local thermal equilibrium state in which the electron temperature is much larger than the lattice temperature, and the electron gas average momentum and energy flux vanish. In the extended hydrodynamical model of Anile and Muscato [11], the first thirteen moment equations are considered and the closure, at the level of the flux of energy flux, is achieved by imposing compliance with the entropy principle up to second order around the state of local thermal equilibrium (supplemented by the maximum entropy ansatz for the distribution function in order to calculate one of the coefficients in the expansion, which otherwise would have remained undetermined [37], [38]). This closure has been checked by Monte Carlo simulation in several cases. The resulting system of equations is hyperbolic in a broad domain of variables. In particular the heat flux obeys an evolution equation of the Maxwell-Cattaneo type (and likewise the viscous stresses, which however are unimportant in this context and will be ignored). In the case of small gradients (of temperature, velocity, etc.) one can linearize the extended system around the state of local thermal equilibrium (the so-called Maxwellian iteration [39]), thereby obtaining a constitutive law for the heat flux . It turns out that the constitutive law for the heat flux comprises a convective term and a conduction one of the Fourier type. The resulting system of evolution equations for particle number, momentum and energy, closed with the above constitutive law for the heat flux, form a parabolic system which is consistent with the entropy principle of linear irreversible thermodynamics and the Onsager reciprocity relations. This system has been amply investigated by various numerical techniques [40].

This parabolic system should be contrasted with the extended hydrodynamical system of above, which is hyperbolic, and which comprises evolution equations for particle number, momentum, energy and an evolution equation for the heat flux. For both systems the relaxation times, which determine the transport coefficients and the production terms (arising from the collisions of electrons with the lattice phonons), have been modeled in the same way and checked by Monte Carlo simulation in a homogeneous slab of silicon. As a benchmark both systems have been solved numerically for the submicron diode consisting of a 0.1 μm n^+ region followed by a 0.4 μm n region, and ending with a 0.1 μm n^+ region. The doping density in the n^+ region is $N = 5 \times 10^7 \ cm^{-3}$, while in the n region it is $N = 2 \times 10^{15} cm^{-3}$. The ambient device temperature is $T_L = 300$ Kelvin. We remark that the diode under consideration presents very steep density gradients at the source and drain junctions and also very high electric fields ($\geq 10^5 V/cm$), which stretches up to its limits any continuum description. Therefore this is an ideal benchmark for testing continuum models against kinetic simulations. The boundary conditions for the two models are consistent. The results for the velocity profile have been compared with Monte Carlo simulations with exactly the same conditions. It is apparent that the hyperbolic models, even in stationary situation, perform much better than the corresponding parabolic one, vis a vis of the Monte Carlo simulations. Also for the velocity profile there is a spike in the parabolic model near the second junction, which is absent in the hyperbolic model (and also in the "true" Monte Carlo simulation -see Figure 2). The spurious unphysical result highlights the difference between the parabolic and hyperbolic model in regions of high electric fields and large gradients.

IV. RELATIVISTIC EXAMPLES

Hyperbolic theories are conceptually more crucial in relativistics scenarios as the "action at a distance" implied by parabolic theories finds no room there.

1. *Precursor peak in astrophysical X-ray bursters*
Many fluids of astrophysical interest fall in the category of "radiative plasmas". By such it is understand an interacting mixture of a material gas (e.g. ionized hydrogen atoms plus electrons) and radiation quanta (e.g. photons). The material medium is considered to be locally in thermal equilibrium with itself as its internal mean free times are extremely short, whereas the radiation quanta, being out of equilibrium with the material medium (its mean free time is finite) gives to rise dissipation. Obviously photons are scattered, absorbed and emitted by the gas [41].

The analysis of elastic photon diffusion through a cloud of ionized plasma of modest optical depth via

parabolic theories predicts for the distribution of the photons over their time of escape from the cloud the conventional diffusion structure, but it is unable to account for the double peaked temporal luminosity profiles observed from X-ray bursters. Hoffman *et al* [42] reported on a X-ray burster showing a precursor peak, that lasted about 4 seconds, neatly separated from the main event which lasted above 1000 seconds. This pattern is reminiscent of those encountered in heat pulse experiments in solids alluded before, it can be explained by the hyperbolic theory of photon transport [43]. The starting equations are

$$\dot{N} + J^a_{,a} = S$$

and

$$\kappa_T J^a + \dot{J}^a + \frac{1}{3} h^{ab} N_{,b} = 0$$

with $N^a (= N u^a + J^a)$ the photon four-current, u^a the rest-frame four-velocity of the material medium, $N = -N^a u_a$ the photon number density, $J^a = h^{ab} N_b$ the convective photon current relative to the ionized plasma, $h^{ab} = g^{ab} + u^a u^b$ the spatial projector, S the photon source (assumed Gaussian in time), and $\kappa_T = n\sigma_T$ where n is the electron number density and σ_T the Thomson cross section. The solution of the corresponding dispersion relation is given in terms of a Green function showing a diffusive behavior for times longer than $t_D \sim \tau *^2 t_{col}/2$ (the conventional diffusion time), and a well defined peak for times shorter than t_D. Here $\tau*$ stands for the optical depth. As seen in figures 2, 3, 5, 6 and 7 of Ref [43] the solution is similar to that found in heat pulses in solids.

2. *Thermal relaxation in gravitational collapse*

Very often in the analysis of the gravitational collapse of fluid spheres the thermal relaxation time τ is ignored [44] since it is usually very small as compared with the typical time scales of collapse for gravitating systems (for phonon-electron interaction $\tau \sim 10^{-11}$ sec, and for phonon-phonon and free electron interaction $\tau \sim 10^{-13}$ sec at room temperature [45]). However, there are situations in which τ cannot be neglected against the gravitational collapse. For instance, in cores of evolved stars the quantum cells of phase space are filled up and the electron mean free-path increases substantially and so does τ. For a completely degenerate core star of radius one hundred of the solar radius and temperature of about ten million Kelvin one has $\tau \sim 1$ sec [14]. In recent works Di Prisco *et al* [46] and Herrera and Martínez [47] using the thermal conductivity by electrons in neutron star matter and the Maxwell-Cattaneo transport equation for the heat flux, showed that the outcome of the

collapse of the star is very sensitive to the value of τ (they restricted themselves to the very conservative range 10^{-6} sec $< \tau < 10^{-3}$ sec), to the point that a fluid sphere may bounce if τ is shorter than some critical value within the mentioned range, otherwise it undergoes a complete gravitational collapse -see figure 4 of Ref. [47]. It is worthy of note that this analysis also applies (modulo some obvious adaptation) to the Kelvin-Helmholtz phase of the birth of a neutron star whose duration is about tens of seconds and in which most of the binding energy (about 10^{53} erg) is emitted [48].

Further, phenomenae prior to relaxation may influence in a important way the subsequent evolution of the system (i.e., for times longer than τ). In this connection it has been shown that for spherically symmetric stars with a radial heat flow, the temperature gradient appearing as a result of perturbations, and thereby the luminosity, are highly dependent on the product of the relaxation time by the period of oscillation of the star [49].

3. *Isotropic early Universe expansion*

It is natural to think that the fluid filling the primeval Universe was subject to steep gradients, whereby the very early cosmic world should be a right scenario for hyperbolic theories of dissipation to apply together with Einstein's field equations. As we will see by resorting to a couple of examples the predictions of the causal theories greatly differ from that of the acausal (parabolic) ones. Unfortunately there is no compelling observational evidence favoring one or another, as nowadays cosmic observation is still at its infancy. As a first example let us consider a homogeneous and isotropic universe with geometry described by a flat Robertson-Walker metric with scale factor $a(t)$. And assume that the source of the metric is a barotropic fluid (i.e., one with equation of state $P = (\gamma - 1)\rho$, $\gamma = $ constant > 0), endowed with a nonvanishing bulk viscosity whose coefficient depends on the energy density via $\zeta = \alpha \rho^m$ with α a positive semidefinite constant and m a constant parameter. By using in this setting the parabolic Eckart's theory of irreversible phenomena Barrow [50] found the first order differential equation for the evolution of the Hubble parameter

$$2\dot{H} + 3(\gamma - 3^m \alpha H^{2m-1})H^2 = 0 , \quad (H \equiv \frac{\dot{a}}{a}) .$$

For $m < 1/2$ this equation predicts, among other things, that the Universe begins with a Friedmann singularity and approaches a inflationary expansion (i.e., $H = $ constant) for $t \rightarrow \infty$. However, if $m > 1/2$ one has what it is termed as a "deflationary" behavior, i.e., the Universe starts with an inflationary expansion and evolves toward a Friedmann state.

Things are different when a hyperbolic theory is used instead. In such a case an expression for the relaxation time must be postulated. Pavón et al [51] choose the well motivated relationship $\tau = \zeta / \rho$, later justified by Maartens [52]. The corresponding differential equation for the evolution of the Hubble factor

$$\alpha \beta^n H^{2n} \ddot{H} + (1 + 3\alpha \beta^n \gamma H^{n+m}) \dot{H}$$
$$+ \left(\frac{3}{2} \gamma - \frac{12\pi\alpha}{\beta^m} H^{n+m} \right) H^2 = 0 \, ,$$

becomes second order in time, and so the number of possible evolutions greatly augments. (Here $n \equiv m - 1$, β is a short hand for $3/(8\pi)$, and units have been chosen so that $c = G = 1$). Last equation reduces to the former one under the unphysical assumption of vanishing τ, and in the very particular case that $\ddot{H} = -3\gamma H \dot{H}$. The set of solutions of the causal equation is far richer than that of the acausal equation, both for nearly stationary solutions and for the nonstationary ones -see figures 1-5 in Ref [51]. Here we want to stress that the deflationary solutions found by Barrow [50] for $m > 1/2$ are seen to be unstable. Initial inflationary expansions can be found only by fine-tunining the initial conditions $H(0)$ and $\dot{H}(0)$, hence it is not a general feature of cosmic evolution. Another point of interest refers to the "generalized second law of thermodynamics", which enters into play when the spacetime posses an event horizon. In such cases is not the entropy of the cosmic fluid that musn't decrease with time ($\dot{S}_f \geq 0$), but the combined entropy of the event horizon S_H and that of the fluid, i.e., $\dot{S}_H + \dot{S}_f \geq 0$. The conceptual trouble with theories allowing unbounded speeds for dissipative signals is that no meaningful event horizon can be ascribed to the spacetime even though it experiences a de Sitter expansion. So one is forced also in this regard to resort to hyperbolic (causal) theories if one wishes to deal with cosmic horizons when the source of the gravitational field is a dissipative fluid. The concrete expressions for the generalized second law, equation (3.53) of Ref [50] and equation (53) of Ref [51], largely differ from one another.

4. *Anisotropic early Universe expansion*
The above study has been extended to anisotropic spaces, Bianchi-type I [53] and Bianchi-type III [54], which admit shear stresses. In both instances the dynamical equations turned out far too complicated to allow analytical integration, so qualitative analysis and numerical integration were carried out instead. For Bianchi-type I the initial anisotropy dies away quickly and, in general, neither the Friedmann nor de Sitter expansions are stable. Nevertheless, for a wide range of values of the parameters occurring in these equations, there exists a stable submanifold of phase space where the latter presents asymptotic stability. This contrasts with the findings of Huang [55], who used Eckart's theory of dissipative processes [4.a] for the bulk and shear stresses, and concluded that the Bianchi-type I space asymptotically evolves either to Friedmann or de Sitter states. For Bianchi-type III the Friedmann expansion is unstable and the de Sitter expansion stable [54].

V. CONCLUDING REMARKS

By resorting to some specific examples in nonrelativistic as well as in relativistic nonequilibrium thermodynamic situations we have made explicit the fact that hyperbolic (causal) theories of transport have a range of applicability much broader than the parabolic (acausal) ones. The differences between both sets may be ignored in steady regimes only, i.e., when the relaxation times of the processes under consideration are much shorter than the time scale of interest. This has been known long since. It was originally hinted by Maxwell [56], and first seen in the laboratory by Peshkov [17] and later confirmed many times over in variety of experimental situations [21], [3]. In this way we have dispelled the claims recently made in [6] and [7] implying that hyperbolic theories have no useful applications. Note that in any case the findings in [6] and [7] are limited to simple fluids and situations such that the quantity Γ, defined above is much lower than unity. Thereby it does not cover many interesting physical fluid situations, such as those recalled in this article. Therefore claims such as *hyperbolic, relativistic theories for dissipative fluids, reduce, physically, to Navier-Stokes* found in [6] cannot be justified. In summary, the suggestion in [6] and [7] that the class of solutions to which their theorems apply include all solutions of the hyperbolic dissipative fluid theories having physical relevance, is at variance with experimental evidence.

Current hyperbolic theories are not free, however, of some drawbacks, which will be mentioned in the following. This does not imply that one should go back to the acausal parabolic theories, but instead try to improve on the existing hyperbolic models. The main drawback of the moment methods currently used in present day hyperbolic theories is the introduction of spurious characteristic speeds, i.e., characteristic waves that arise only from the truncation procedure (the number of moments retained in the theory), and have no counterpart in the underlying kinetic description. Since these characteristics constitute a fan which broadens as the number of moments increases, there could be serious difficulties (due to the mathematical properties of the hyperbolic systems) of the appearance of unphysical discontinuities in the solutions of the hyperbolic theories at places where the fluid speed crosses one of the characteristic speeds [57].

Fortunately the problem is not so severe because lately Weiss [58] has shown, by direct numerical integration (for the case of a rarefied gas of Maxwell molecules) that, by some algebraic cancellation, the discontinuities can occur only at the highest characteristic speed, and the latter increases with the number of retained moments. If this result would be confirmed (and mathematically understood) in more general cases, the aforesaid difficulty would be alleviated.

More generally, the root of the difficulty lies in using the usual polynomial moments. In fact an avenue for further research should be to investigate whether a different basis function would be more suitable. We remark that any basis function must be able to capture phenomena varying on a very short time scale. In fact, the requirement of relativistic causality itself, forces the basis function to be capable of describing high frequency situations. One formulation of relativistic causality is that the wave speeds must be less or equal to the light speed in vacuum c. This must be applied to the characteristic speeds, but also to the dispersion waves arising from a linearization of the equations. In this case the relativistic causality takes the form $v_g \leq c$ as $\omega \to \infty$ (where v_g is the group velocity), and obviously this requires that the equations must be able to describe high frequency phenomena (at least in the asymptotic sense). This is the main physical reason why quasi-stationary theories, based on slowly varying variables, are intrinsically unable to comply with relativistic causality and makes it mandatory to consider hyperbolic theories.

VI. ACKNOWLEDGMENTS

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TABLE I

l/λ	0.25	0.50	1.00	2.00	4.00	7.00
$(c_s/v_{ph})_{FNS}$	0.40	0.26	0.19	0.13	0.10	0.07
$(c_s/v_{ph})_{Hyp.}$	0.52	0.43	0.44	0.47	0.48	0.49
$(c_s/v_{ph})_{Exp.}$	0.51	0.46	0.50	0.46	0.46	0.46

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Table I
Numerical values of c_s/v_{ph} as a function of l/λ.

LIST OF CAPTIONS FOR FIGURES

Figure 1
Schematic results of heat pulse experiments. The dashed line represents the heat pulse at the source, whereas the solid line shows the structure of the heat signal at an arbitrarily selected point in in the sample. Peaks **A**, **B** and **C** correspond to ballistic phonons with longitudinal sound velocity, ballistic phonons with transverse sound speed, and second sound, respectively. The tail **D** corresponds to the diffusive heat propagation. This structure can be explained by hyperbolic theories. By contrast, parabolic theories are able to explain the tail **D** only.

Figure 2
Velocity profile for the electrons in a submicron diode under a very high electric field. The crosses correspond to the Monte Carlo simulation, the solid line to the prediction of the hyperbolic theory, and the dashed line to the parabolic one. The latter shows a spurious peak.